Solving a system of linear equations by Cramer's rule

A system of three linear equations in three variables can be solved by using determinants in a manner similar to that of the previous section. This rule is also called Cramer's rule. The solution to the system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Is given by

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \quad \text{and} \quad z = \frac{D_z}{D}$$

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$,

and $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, provided that $D \neq 0$

Note that D_x, D_y and D_z are obtained from D by replacing the $x-, y-$ or $z-$ column with the constants d_1, d_2 and d_3.

Example 1: Use Cramer's rule to solve the system

$$x + 2y - 3z = -2, \quad x - y + z = -1 \text{ and } 3x + 4y - 4z = 4$$

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 1 & -1 & 1 \\ 3 & 4 & -4 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -7$$

$$D_x = \begin{vmatrix} -2 & 2 & -3 \\ -1 & -1 & 1 \\ 4 & 4 & -4 \end{vmatrix} = (-2)\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} + (-3)\begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 1 & -2 & -3 \\ 1 & -1 & 1 \\ 3 & 4 & -4 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -35$$

$$D_z = \begin{vmatrix} 1 & 2 & -2 \\ 1 & -1 & -1 \\ 3 & 4 & 4 \end{vmatrix} = 1\begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = -28$$

$$x = \frac{D_x}{D} = \frac{0}{-7} = 0, \quad y = \frac{D_y}{D} = \frac{-35}{-7} = -5 \quad \text{and} \quad z = \frac{D_z}{D} = \frac{-28}{-7} = 4$$

Example 2:Use Cramer's rule to solve the system

$2x + 3y - z = 1$, $4x + y - 3z = 11$ and $3x - 2y + 5z = 21$□

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & -3 \\ 3 & -2 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 4 & -3 \\ 3 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -78$$□

$$D_x = \begin{vmatrix} 1 & 3 & -1 \\ 11 & 1 & -3 \\ 21 & -2 & 5 \end{vmatrix} = 1\begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 11 & -3 \\ 21 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 11 & 1 \\ 21 & -2 \end{vmatrix} = -312$$□

$$D_y = \begin{vmatrix} 2 & 1 & -1 \\ 4 & 11 & -3 \\ 3 & 21 & 5 \end{vmatrix} = 2\begin{vmatrix} 11 & -3 \\ 21 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & -3 \\ 3 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 11 \\ 3 & 21 \end{vmatrix} = 156$$□

$$D_z = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 11 \\ 3 & -2 & 21 \end{vmatrix} = 2\begin{vmatrix} 1 & 11 \\ -2 & 21 \end{vmatrix} - 3\begin{vmatrix} 4 & 11 \\ 3 & 21 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -78$$□

$$x = \frac{D_x}{D} = \frac{-312}{-78} = 4 \quad , \quad y = \frac{D_y}{D} = \frac{156}{-78} = -2 \quad \text{and} \quad z = \frac{D_z}{D} = \frac{-78}{-78} = 1$$

Exercises

Use Cramer's rule to solve the system

(1) $x + 2y + 3z = 17$

$3x + 2y + z = 11$

$x - 5y + z = -5$ □

(2) $x + 2y - 3z = -2$

$x - y + z = -1$

$3x + 4y - 4z = 4$